GCE A LEVEL MARKING SCHEME

SUMMER 2019

A LEVEL (NEW)
FURTHER MATHEMATICS
UNIT 4 FURTHER PURE MATHEMATICS B
1305U40-1

INTRODUCTION

This marking scheme was used by WJEC for the 2019 examination. It was finalised after detailed discussion at examiners' conferences by all the examiners involved in the assessment. The conference was held shortly after the paper was taken so that reference could be made to the full range of candidates' responses, with photocopied scripts forming the basis of discussion. The aim of the conference was to ensure that the marking scheme was interpreted and applied in the same way by all examiners.

It is hoped that this information will be of assistance to centres but it is recognised at the same time that, without the benefit of participation in the examiners' conference, teachers may have different views on certain matters of detail or interpretation.

WJEC regrets that it cannot enter into any discussion or correspondence about this marking scheme.

GCE FURTHER MATHEMATICS

A2 UNIT 4 FURTHER PURE MATHEMATICS B

SUMMER 2019 MARK SCHEME

1. a)	$\lvert z\rvert = \sqrt{3^2+4^2} = 5$ $arg(z) = \tan^{-1}\frac{4}{3} = 0.93$ $\therefore z = 5e^{0.93i}$	B1 B1 B1	 Condone degrees Radian form
b) i)	$\sqrt[3]{z} = \sqrt[3]{5}e^{0.31i+\frac{2n\pi i}{3}}$ $z_1 = \sqrt[3]{5}e^{0.31i} = 1.63 + 0.52i \ \ (1.63, 0.52)$ $z_2 = \sqrt[3]{5}e^{2.40i} = -1.26 + 1.15i \ \ (-1.26, 1.15)$ $z_3 = \sqrt[3]{5}e^{4.50i} = -0.36 - 1.67i \ \ (-0.36, -1.67)$	M1 A1 m1 A1	FT (a) A1 complex form for 1 root $+\frac{2n\pi i}{3}$ A1 for all correct coordinates
ii)	Equilateral (triangle)	B1	
2. a)	$3\left(\frac{2t}{1+t^2}\right) + 4\left(\frac{1-t^2}{1+t^2}\right) - 2$ $3\left(\frac{2t}{1+t^2}\right) + 4\left(\frac{1-t^2}{1+t^2}\right) - 2\left(\frac{1+t^2}{1+t^2}\right)$ $= \frac{6t+2-6t^2}{1+t^2}$	M1 A1	 Convincing
b)	$\frac{6t+2-6t^2}{1+t^2} = 3$ $6t + 2 - 6t^2 = 3 + 3t^2$ $9t^2 - 6t + 1 = 0$ Solving, eg $(3t-1)^2 = 0$ $t = \frac{1}{3}$ $\therefore \tan\frac{1}{2}x = \frac{1}{3}$ $\frac{1}{2}x = 18.43^o + 180^o n$ $x = 36.87^o + 360^o n$	M1 A1 m1 A1 M1 A1 A1	 m0 no working Attempt to solve for x. FT their t Accept radians: $0.322 + n\pi$ $0.644 + 2n\pi$
3. a)	METHOD 1: $det = 2(12+16) + 7(0-14) + 2(0+21) = 0$ $\therefore$ the equations do not have a unique solution METHOD 2: Reduction to echelon form Correct matrix to determine nature of solutions, eg $\left(\begin{array}{ccc\|c} 2 & -7 & 2 & a \\ 0 & 33 & -22 & 11b \\ 0 & 33 & -22 & -7a-2c \end{array}\right)$	 M1A1 E1 (M1) (A1)	 FT det

	Correct statement eg No unique solutions when $11b = -7a - 2c$	(E1)	
b)	METHOD 1: $\left(\begin{array}{ccc\|c} 1 & 8 & -6 & 5 \\ 2 & 4 & 6 & -3 \\ -5 & -4 & 9 & -7 \end{array}\right)$	M1	
	By row operations: $\left(\begin{array}{ccc\|c} 1 & 8 & -6 & 5 \\ 0 & -12 & 18 & -13 \\ 0 & 36 & -21 & 18 \end{array}\right)$	A1	Reduction to $\begin{pmatrix} 1 & & \\ 0 & & \\ 0 & & \end{pmatrix}$
	Then $\left(\begin{array}{ccc\|c} 1 & 8 & -6 & 5 \\ 0 & -12 & 18 & -13 \\ 0 & 0 & 33 & -21 \end{array}\right)$	A1	Reduction to $\begin{pmatrix} 1 & & \\ 0 & & \\ 0 & 0 & \end{pmatrix}$
	Therefore $z = \frac{-7}{11} \qquad y = \frac{17}{132} \qquad x = \frac{5}{33}$ cao	A2	A1 for any 2 correct
	METHOD 2: Rearranging $x = 5 - 8y + 6z$ and Substituting:	(M1)	
	2nd equation: $-12y + 18z = -13$ 3rd equation: $36y - 21z = 18$	(A1)	
	Solving,	(m1)	
	$z = \frac{-7}{11} \qquad y = \frac{17}{132} \qquad x = \frac{5}{33}$ cao	(A2)	A1 for any 2 correct
	METHOD 3: Let $A = \begin{pmatrix} 1 & 8 & -6 \\ 2 & 4 & 6 \\ -5 & -4 & 9 \end{pmatrix}$ and $B = \begin{pmatrix} 5 \\ -3 \\ -7 \end{pmatrix}$ $det = 1(36 + 24) - 8(18 + 30) - 6(-8 + 20) = -396$		
	$A^{-1} = \frac{1}{-396}\begin{pmatrix} 60 & -48 & 72 \\ -48 & -21 & -18 \\ 12 & -36 & -12 \end{pmatrix}$	(M1) (A1)	M0 no working
	Therefore, $X = A^{-1}B$ $X = \frac{1}{-396}\begin{pmatrix} 60 & -48 & 72 \\ -48 & -21 & -18 \\ 12 & -36 & -12 \end{pmatrix}\begin{pmatrix} 5 \\ -3 \\ -7 \end{pmatrix}$	(m1)	
	$X = \begin{pmatrix} 5/33 \\ 17/132 \\ -7/11 \end{pmatrix}$ cao	(A2)	A1 for any 2 correct

4. a)	$x = \cot y$ $\frac{dx}{dy} = -\operatorname{cosec}^2 y$ $\frac{dx}{dy} = -(\cot^2 y + 1)$ $\frac{dx}{dy} = -(x^2 + 1)$ $\frac{dy}{dx} = \frac{-1}{x^2 + 1}$	M1 A1 A1 A1 A1	 Convincing
b)	$\frac{6x^2 - 10x - 9}{(2x+3)(x^2+1)} = \frac{A}{2x+3} + \frac{Bx + C}{x^2+1}$ $6x^2 - 10x - 9 = A(x^2 + 1) + (Bx + C)(2x + 3)$ When $x = -1.5$, $19.5 = 3.25A$ $\therefore A = 6$ When $x = 0$, $-9 = A + 3C$ $\therefore C = -5$ Co-efficients of x^2: $6 = A + 2B$ $\therefore B = 0$ $\frac{6x^2 - 10x - 9}{(2x+3)(x^2+1)} = \frac{6}{2x+3} - \frac{5}{x^2+1}$	M1 A1 A1 A1 A1	Use of
c)	$\frac{6x^2 - 8x - 6}{(2x+3)(x^2+1)}$ $= \frac{6}{2x+3} - \frac{5}{x^2+1} + \frac{2x+3}{(2x+3)(x^2+1)}$ $= \frac{6}{2x+3} - \frac{4}{x^2+1}$ $\therefore \int \left(\frac{6}{2x+3} - \frac{4}{x^2+1}\right) dx$ $= 3\ln\|2x+3\| + 4\cot^{-1} x + c$	M1 A1 M1 A1A1	FT (b) Accept $-4\tan^{-1} x$ Penalise -1 for no constant term
d)	The expression is undefined when $x = -1.5$ (and -1.5 is in the range of integration.)	E1	

5. a)	$\sin\theta - \sin 3\theta = 2\cos\left(\frac{\theta+3\theta}{2}\right)\sin\left(\frac{\theta-3\theta}{2}\right)$ $= 2\cos 2\theta \sin(-\theta)$ $= -2\cos 2\theta \sin(\theta)$	M1 A1 A1	
b)	$y = 2\cos 2\theta \sin\theta + 7 = -\sin\theta + \sin 3\theta + 7$ Mean value = $\frac{1}{2}\int_1^3(-\sin\theta + \sin 3\theta + 7)d\theta$ $= \frac{1}{2}\left[\cos\theta - \frac{1}{3}\cos 3\theta + 7\theta\right]_1^3$ $= \frac{1}{2}\left(\cos 3 - \frac{1}{3}\cos 9 + 7\times 3\right) - \frac{1}{2}\left(\cos 1 - \frac{1}{3}\cos 3 + 7\right)$ $= 6.22$	B1 M1 A1 m1 A1	FT (a) Use of limits If M0, award SC1 for 6.22 unsupported
6.	Auxiliary equation $r^2 - 7r + 10 = 0$ Roots $r = 2$ and $r = 5$ General solution $y = Ae^{5x} + Be^{2x}$ Differentiating, $\frac{dy}{dx} = 5Ae^{5x} + 2Be^{2x}$ $\frac{d^2y}{dx^2} = 25Ae^{5x} + 4Be^{2x}$ Substituting. $1 = 5A + 2B$ $8 = 25A + 4B$ Solving, $A = 2/5$ or $B = -1/2$ oe $\therefore\ y = \frac{2}{5}e^{5x} - \frac{1}{2}e^{2x}$	M1 A1 B1 M1 A1 A1 M1 A1 A1 A1	1 correct value

7. a)	Use of formula booklet expansion with $-x$ $\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3}$	M1 A1	
b)	METHOD 1: $-2\ln\left(\frac{1-x}{(1+x)^2}\right) = -2[\ln(1-x) - 2\ln(1+x)]$ $= -2 \times \left[\left(-x - \frac{x^2}{2} - \frac{x^3}{3}\right) - 2\left(x - \frac{x^2}{2} + \frac{x^3}{3}\right)\right]$ $= -2 \times \left[-3x + \frac{x^2}{2} - x^3\right]$ $= 6x - x^2 + 2x^3$	M1A1 A1 A1	 FT their (a) cao Mark final answer
	METHOD 2: Let $f(x) = -2\ln\left(\frac{1-x}{(1+x)^2}\right)$ $\{= -2\ln(1-x) + 4\ln(1+x)\}$ $f(0) = 0$ $f'(x) = \frac{2}{1-x} + \frac{4}{1+x}$ $f'(0) = 6$ $f''(x) = 2(1-x)^{-2} - 4(1+x)^{-2}$ $f''(0) = -2$ $f'''(x) = 4(1-x)^{-3} + 8(1+x)^{-3}$ $f'''(0) = 12$ Therefore, $-2\ln\left(\frac{1-x}{(1+x)^2}\right) = 0 + x(6) + x^2\left(-\frac{2}{2!}\right) + x^3\left(\frac{12}{3!}\right)$ $-2\ln\left(\frac{1-x}{(1+x)^2}\right) = 6x - x^2 + 2x^3$	 (M1) (A3)	 cao A1 each correct term Mark final answer

8. a)	$y = r\sin\theta = \sin(2\theta)\sin\theta$ $\frac{dy}{d\theta} = \sin(2\theta)\cos\theta + 2\sin\theta\cos(2\theta)$ When parallel to initial line: $\frac{dy}{d\theta} = \sin(2\theta)\cos\theta + 2\sin\theta\cos(2\theta) = 0$ METHOD 1: $\sin(2\theta)\cos\theta = -2\sin\theta\cos(2\theta)$ $\tan(2\theta) = -2\tan\theta$ $\frac{2\tan\theta}{1-\tan^2\theta} = -2\tan\theta$ $2\tan^3\theta - 4\tan\theta = 0$ $2\tan\theta\,(\tan^2\theta - 2) = 0$ $\tan\theta = \sqrt{2}$ (or $\tan\theta = 0$ or $\tan\theta = -\sqrt{2}$) Therefore, $\theta = 0.955$ $r = 0.943$ METHOD 2: $2\sin\theta\cos^2\theta + 2\sin\theta\,(2\cos^2\theta - 1) = 0$ $2\sin\theta\,(3\cos^2\theta - 1) = 0$ $\cos\theta = +\frac{1}{\sqrt{3}}$ (or $\sin\theta = 0$ or $\cos\theta = -\frac{1}{\sqrt{3}}$) Therefore, $\theta = 0.955$ $r = 0.943$	M1 A2 m1 m1 A1 A1 A2 (m1) (A1) (A1) (A2)	 A1 each term $\frac{dy}{d\theta} = 0$ Use of double-angle formulae Solveable form A1 each part FT their $\tan\theta$ provided θ in range and M1m1m1 Use of double-angle formulae Solveable form A1 each part FT their $\cos\theta$ provided θ in range and M1m1m1
b)	(0.544, 0.770)	B1	FT (a)

<table>
<tr><td>9.
a)</td><td>METHOD 1:
$\frac{dy}{d\theta} = \frac{1}{\sqrt{1-(\cos\theta)^2}} \times -\sin\theta$
$\frac{dy}{d\theta} = \frac{1}{\sin\theta} \times -\sin\theta$
$\frac{dy}{d\theta} = -1$

METHOD 2:
$\sin y = \cos\theta$
$\cos y \frac{dy}{d\theta} = -\sin\theta$
$\frac{dy}{d\theta} = \frac{-\sin\theta}{\cos y}$
$\cos^2 y = 1 - \sin^2 y$
$\therefore \cos y = \sqrt{1-\sin^2 y} = \sqrt{1-\cos^2\theta} = \sin\theta$
$\frac{dy}{d\theta} = \frac{-\sin\theta}{\sin\theta} = -1$</td><td>M1A1

A1

A1

(M1)

(A1)

(A1)

(A1)</td><td></td></tr>
<tr><td>b)</td><td>Differentiating
$\frac{dy}{dx} = x^3 \times \frac{1}{1+(4x)^2} \times 4 \quad + \quad 3x^2 \times \tan^{-1}(4x)$

When $x = \frac{\pi}{2}, \frac{dy}{dx} = 10.8(42)$</td><td>M1
A1A1

A1</td><td>
A1 for each term</td></tr>
<tr><td>c)</td><td>METHOD 1:
$\frac{dy}{dx} = \frac{1}{1-(1-x)^2} \times -1$

METHOD 2:
$\tanh y = 1 - x$
$\operatorname{sech}^2 y \frac{dy}{dx} = -1$
$\frac{dy}{dx} = \frac{-1}{\operatorname{sech}^2 y} = \frac{-1}{1-\tanh^2 y} = \frac{-1}{1-(1-x)^2}$

THEN:
When $x = 1.7, \frac{dy}{dx} = -\frac{100}{51}$ and $y = -0.8673$
Gradient of normal = $\frac{51}{100}$
Equation: $y + 0.8673 = \frac{51}{100}(x - 1.7)$</td><td>M1A1

(M1)

(A1)

B1B1
B1
B1</td><td>

B1 for each part
FT value of $\frac{dy}{dx}$
cao</td></tr>
</table>

10.	Dividing both sides by $\sec x$: $\frac{dy}{dx} + \frac{y\cos x}{\sin x} = 2\cos x$ Integrating factor: $e^{\int \frac{\cos x}{\sin x} dx}$ $= e^{\ln \sin x} = \sin x$ Multiplying both sides: $\sin x \frac{dy}{dx} + y\cos x = 2\sin x\cos x \quad (= \sin 2x)$ Integrating: $y\sin x = \frac{-\cos 2x}{2} + c$ Substituting $3 = \frac{1}{2} + c \quad \to \quad c = \frac{5}{2}$ Solution: $y\sin x = \frac{-\cos 2x}{2} + \frac{5}{2}$ When $x = \frac{\pi}{4}$, $y\sin\frac{\pi}{4} = \frac{5}{2} \quad \to \quad y = \frac{5\sqrt{2}}{2} \quad (3.536)$	M1 M1 A1 M1 m1 A1 B1 B1	 Or equivalent
11. a)	Area = $\int_0^1 x\sinh(x)\ dx$ $= [x\cosh x]_0^1 - \int_0^1 \cosh x\,dx$ $=[x\cosh x]_0^1 - [\sinh x]_0^1$ $= 0.368 \ \left(\frac{1}{e}\right)$	 M1A1 A1 A1	 A1 for all correct M0 unsupported answer
b)	Volume = $\pi \int_0^1 \cosh^2(2x)\,dx$ $= \frac{\pi}{2}\int_0^1 (1 + \cosh 4x)dx$ $= \frac{\pi}{2}\left[x + \frac{1}{4}\sinh 4x\right]_0^1$ =12.3	M1 A1 A1 A1	
c)	24.6	B1	FT (b) x 2

12. a)	$\int_3^4 \frac{1}{\sqrt{x^2-4}}dx = \left[\cosh^{-1}\frac{x}{2}\right]_3^4$ $= 0.355$	M1A1 A1	M0 unsupported answer
b)	$\int_1^2 \frac{k}{9-x^2}dx = \left[\frac{k}{6}\ln\left\|\frac{3+x}{3-x}\right\|\right]_1^2$ $= \frac{k}{6}(\ln 5 - \ln 2) \qquad \left(= \frac{k}{6}\ln\frac{5}{2}\right)$ METHOD 1: $\ln\frac{25}{4} = \ln\left(\frac{5}{2}\right)^2 = 2\ln\frac{5}{2}$ Equating, $\therefore\ \frac{k}{6}\ln\frac{5}{2} = 2\ln\frac{5}{2}$ $k = 12$ METHOD 2: Equating: $\frac{k}{6}(\ln 5 - \ln 2) = \ln\frac{25}{4}$ $\frac{k}{6} = 2$ $k = 12$	M1A1 A1 m1 A1 (m1) (A1)	
c)	METHOD 1: $\sinh^2 x + \cosh^2 x - \sinh 2x$ $= \sinh^2 x + \cosh^2 x - 2\sinh x\cosh x$ $= (\cosh x - \sinh x)^2$ $\int \frac{(\cosh x - \sinh x)^3}{(\cosh x - \sinh x)^2}dx = \int (\cosh x - \sinh x)\,dx$ $= \sinh x - \cosh x + c$ $= \frac{e^x - e^{-x}}{2} - \frac{e^x + e^{-x}}{2} + c$ $= -e^{-x} + c$ METHOD 2: $\sinh^2 x + \cosh^2 x - \sinh 2x$ $= \sinh^2 x + \cosh^2 x - 2\sinh x\cosh x$ $= (\cosh x - \sinh x)^2$ $\int \frac{(\cosh x - \sinh x)^3}{(\cosh x - \sinh x)^2}dx = \int \cosh x - \sinh x\,dx$ $= \int \left(\frac{e^x + e^{-x}}{2} - \frac{e^x - e^{-x}}{2}\right)dx$ $= \int e^{-x}dx$ $= -e^{-x} + c$	 M1 A1 A1 A1 A1 A1 (M1) (A1) (A1) (A1) (A1) (A1)	 Convincing Convincing

	METHOD 3: $\cosh x - \sinh x = \frac{e^x + e^{-x}}{2} - \frac{e^x - e^{-x}}{2} = e^{-x}$	(B1)	Simplified fully
	$\cosh^2 x = \left(\frac{e^x + e^{-x}}{2}\right)^2 = \frac{e^{2x} + e^{-2x} + 2}{4}$ $\sinh^2 x = \left(\frac{e^x - e^{-x}}{2}\right)^2 = \frac{e^{2x} + e^{-2x} - 2}{4}$	(B1)	For either $\cosh^2 x$ or $\sinh^2 x$ simplified
	$\cosh^2 x + \sinh^2 x - \sinh 2x$ $= \frac{e^{2x} + e^{-2x} + 2}{4} + \frac{e^{2x} + e^{-2x} - 2}{4} + \frac{e^{2x} - e^{-2x}}{2}$ $= \frac{e^{2x} + e^{-2x} + 2}{4} + \frac{e^{2x} + e^{-2x} - 2}{4} - \frac{2e^{2x} - 2e^{-2x}}{4}$	(B1)	Common denominator
	Therefore, $\int \frac{(\cosh x - \sinh x)^3}{(\cosh x - \sinh x)^2} dx$ $= \int \frac{(e^{-x})^3}{\frac{e^{2x} + e^{-2x} + 2}{4} + \frac{e^{2x} + e^{-2x} - 2}{4} - \frac{2e^{2x} - 2e^{-2x}}{4}} dx$	(M1)	
	$= \int \frac{e^{-3x}}{e^{-2x}} dx$ $= \int e^{-x} dx$ $= -e^{-x} + c$	(A1) (A1)	Convincing